Deformations of the central extension of the Poisson superalgebra.

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Abstract

Poisson superalgebras realized on the smooth Grassmann valued functions with compact support in $\mathbb{R}^n$ have the central extensions. The deformations of these central extensions are found.

1 Introduction

This paper continues the investigation started in [2], [3] and [4]. We consider the Poisson superalgebra realized on smooth Grassmann-valued functions with compact support in $\mathbb{R}^n$. As it is shown in [2] this superalgebra has central extensions for some dimensions. For these dimensions, we found the second adjoint cohomology space and the deformations of the Poisson superalgebra under consideration.

1.1 Poisson superalgebra

Let $\mathbb{K}$ be either $\mathbb{R}$ or $\mathbb{C}$. We denote by $\mathcal{D}(\mathbb{R}^n)$ the space of smooth $\mathbb{K}$-valued functions with compact support on $\mathbb{R}^n$. This space is endowed with its standard topology. We set

$$D^{n_-}_{n_+} = \mathcal{D}(\mathbb{R}^{n_+}) \otimes \mathbb{G}^{n_-}, \quad E^{n_-}_{n_+} = C^\infty(\mathbb{R}^{n_+}) \otimes \mathbb{G}^{n_-}, \quad D'^{n_-}_{n_+} = \mathcal{D}'(\mathbb{R}^{n_+}) \otimes \mathbb{G}^{n_-},$$

where $\mathbb{G}^{n_-}$ is the Grassmann algebra with n_- generators and $\mathcal{D}'(\mathbb{R}^{n_+})$ is the space of continuous linear functionals on $\mathcal{D}(\mathbb{R}^{n_+})$. The generators of the Grassmann algebra (resp., the coordinates of the space $\mathbb{R}^{n_+}$) are denoted by ξ^α, $\alpha = 1, \ldots, n_-$ (resp., x^i, $i = 1, \ldots, n_+$). We shall also use collective variables z^A which are equal to x^A for $A = 1, \ldots, n_+$ and are equal to ξ^{A-n_+} for $A = n_+ + 1, \ldots, n_+ + n_-$. The spaces $D^{n_-}_{n_+}$, $E^{n_-}_{n_+}$, and $D'^{n_-}_{n_+}$ possess a natural grading which is determined by that of the Grassmann algebra. The parity of an element

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f of these spaces is denoted by $\varepsilon(f)$. We also set $\varepsilon_A = 0$ for $A = 1, \ldots, n_+$ and $\varepsilon_A = 1$ for $A = n_+ + 1, \ldots, n_+ + n_-$. Each function $f \in E^{n_-}_{n_+}$ can be expressed in the form

$$f(z) = \sum_{\alpha_i = 0,1} f_{\alpha_1,\ldots,\alpha_{n_-}}(x)\xi_1^{\alpha_1}\ldots\xi_{n_-}^{\alpha_{n_-}}.$$

We will use the notation $\mathrm{supp}(f) = \bigcup_{\{\alpha_i\}} \mathrm{supp}(f_{\alpha_1,\ldots,\alpha_{n_-}}) \subset \mathbb{R}^{n_+}$.

Let $\partial/\partial z^A$ and $\overleftarrow{\partial}/\partial z^A$ be the operators of the left and right differentiation. The Poisson bracket is defined by the relation

$$\{f, g\}(z) = f(z)\frac{\overleftarrow{\partial}}{\partial z^A}\omega^{AB}\frac{\partial}{\partial z^B}g(z) = -(-1)^{\varepsilon(f)\varepsilon(g)}\{g, f\}(z), \tag{1}$$

where the symplectic metric $\omega^{AB} = -(-1)^{\varepsilon_A\varepsilon_B}\omega^{BA}$ is a constant invertible matrix. For definiteness, we choose it in the form

$$\omega^{AB} = \begin{pmatrix} \omega^{ij} & 0 \\ 0 & \lambda_\alpha\delta^{\alpha\beta} \end{pmatrix}, \quad \lambda_\alpha = \pm 1,\ i, j = 1, ..., n_+,\ \alpha, \beta = 1 + n_+, ..., n_- + n_+,$$

where ω^{ij} is the canonical symplectic form (if $\mathbb{K} = \mathbb{C}$, then one can choose $\lambda_\alpha = 1$). The nondegeneracy of the matrix ω^{AB} implies, in particular, that n_+ is even. The Poisson superbracket satisfies the Jacobi identity

$$(-1)^{\varepsilon(f)\varepsilon(h)}\{f, \{g, h\}\}(z) + \mathrm{cycle}(f, g, h) = 0, \quad f, g, h \in E^{n_-}_{n_+}. \tag{2}$$

By Poisson superalgebra, we mean the space $D^{n_-}_{n_+}$ with the Poisson bracket (1) on it. The relations (1) and (2) show that this bracket indeed determines a Lie superalgebra structure on $D^{n_-}_{n_+}$.

The integral on $D^{n_-}_{n_+}$ is defined by the relation $\bar{f} \stackrel{def}{=} \int dz\, f(z) = \int_{\mathbb{R}^{n_+}} dx \int d\xi\, f(z)$, where the integral on the Grassmann algebra is normed by the condition $\int d\xi\, \xi^1 \ldots \xi^{n_-} = 1$. We identify $\mathbb{G}^{n_-}$ with its dual space $\mathbb{G}'^{n_-}$ setting $f(g) = \int d\xi\, f(\xi)g(\xi)$, $f, g \in \mathbb{G}^{n_-}$. Correspondingly, the space $D'^{n_-}_{n_+}$ of continuous linear functionals on $D^{n_-}_{n_+}$ is identified with the space $\mathcal{D}'(\mathbb{R}^{n_+}) \otimes \mathbb{G}^{n_-}$. As a rule, the value $m(f)$ of a functional $m \in D'^{n_-}_{n_+}$ on a test function $f \in D^{n_-}_{n_+}$ will be written in the "integral" form: $m(f) = \int dz\, m(z)f(z)$.

1.2 Cohomologies of Poisson superalgebra

Let L be a Lie superalgebra acting in a $\mathbb{Z}_2$-graded space V (the action of $f \in L$ on $v \in V$ will be denoted by $f \cdot v$). The space $C_p(L, V)$ of p-cochains consists of all multilinear superantisymmetric mappings from L^p to V. The space $C_p(L, V)$ possesses a natural $\mathbb{Z}_2$-grading: by definition, $M_p \in C_p(L, V)$ has the definite parity $\varepsilon(M_p)$ if

$$\varepsilon(M_p(f_1, \ldots, f_p)) = \varepsilon(M_p) + \varepsilon(f_1) + \ldots + \varepsilon(f_1)$$

for any $f_j \in L$ with parities $\epsilon(f_j)$. The differential d_p^V is defined to be the linear operator from $C_p(L,V)$ to $C_{p+1}(L,V)$ such that

$$d_p^V M_p(f_1,...,f_{p+1}) = -\sum_{j=1}^{p+1}(-1)^{j+\varepsilon(f_j)|\varepsilon(f)|_{1,j-1}+\varepsilon(f_j)\varepsilon_{M_p}} f_j \cdot M_p(f_1,...,\breve{f}_j,...,f_{p+1}) -$$
$$- \sum_{i<j}(-1)^{j+\varepsilon(f_j)|\varepsilon(f)|_{i+1,j-1}} M_p(f_1,...f_{i-1},\{f_i,f_j\},f_{i+1},...,\breve{f}_j,...,f_{p+1}), \quad (3)$$

for any $M_p \in C_p(L,V)$ and $f_1,\dots f_{p+1} \in L$ having definite parities. Here the sign $\breve{}$ means that the argument is omitted and the notation

$$|\varepsilon(f)|_{i,j} = \sum_{l=i}^{j}\varepsilon(f_l)$$

has been used. The differential d^V is nilpotent (see [1]), *i.e.*, $d_{p+1}^V d_p^V = 0$ for any $p = 0,1,\dots$. The p-th cohomology space of the differential d_p^V will be denoted by H_V^p. The second cohomology space H_{ad}^2 in the adjoint representation is closely related to the problem of finding formal deformations of the Lie bracket $\{\cdot,\cdot\}$ of the form $\{f,g\}_* = \{f,g\} + \hbar^2\{f,g\}_1 + \dots$. The condition that $\{\cdot,\cdot\}$ is a 2-cocycle is equivalent to the Jacobi identity for $\{\cdot,\cdot\}_*$ modulo the $\hbar^4$-order terms.

In [2] and [4], we studied the cohomologies of the Poisson algebra $D_{n_+}^{n_-}$ in the following representations:

1. The trivial representation: $V = \mathbb{K}$, $f \cdot a = 0$ for any $f \in D_{n_+}^{n_-}$ and $a \in \mathbb{K}$. The space $C_p(D_{n_+}^{n_-},\mathbb{K})$ consists of separately continuous antisymmetric multilinear forms on $(D_{n_+}^{n_-})^p$. The cohomology spaces and the differentials is denoted by H_{tr}^p and d_p^{tr} respectively.

2. The adjoint representation: $V = D_{n_+}^{n_-}$ and $f \cdot g = \{f,g\}$ for any $f,g \in D_{n_+}^{n_-}$. The space $C_p(D_{n_+}^{n_-}, D_{n_+}^{n_-})$ consists of separately continuous antisymmetric multilinear mappings from $(D_{n_+}^{n_-})^p$ to $D_{n_+}^{n_-}$. The cohomology spaces and the differentials is denoted by H_{ad}^p and d_p^{ad} respectively.

The following theorems are proved in [2] and [4]:

Theorem 1

Let bilinear forms C_2^1 and C_2^2 be defined by the relations

$$C_2^1(f,g) = \bar{f}\bar{g}, \quad (4)$$

$$C_2^2(f,g) = \int dz\left(f(z)\mathcal{E}_z g(z) - (-1)^{\varepsilon(f)\varepsilon(g)} g(z)\mathcal{E}_z f(z)\right), \quad f,g \in D_{n_+}^{n_-}, \quad (5)$$

where $\mathcal{E}_z \overset{def}{=} 1 - \frac{1}{2}z^A\frac{\partial}{\partial z^A}$. [1]

If n_- is even and $n_- \neq n_+ + 4$, then $H_{\mathrm{tr}}^2 = 0$;
if $n_- = n_+ + 4$, then $H_{\mathrm{tr}}^2 \simeq \mathbb{K}$ and the form C_2^2 is a nontrivial cocycle;
if n_- is odd, then $H_{\mathrm{tr}}^2 \simeq \mathbb{K}$ and the form C_2^1 is a nontrivial cocycle.

[1]The operator $\mathcal{E}_z$ is a derivation of the Poisson superalgebra.

Theorem 2 *Let V_1 be the one-dimensional subspace of $C_1(\mathbf{D}^{n_-}_{n_+}, \mathbf{D}^{n_-}_{n_+})$ generated by the cocycle*

$$m_{1|2} = \mathcal{E}_z f(z).$$

Then there is a natural isomorphism $V_1 \oplus (\mathbf{E}^{n_-}_{n_+}/\mathbf{D}^{n_-}_{n_+}) \simeq H^1_{\mathrm{ad}}$ taking $(M_1, T) \in V_1 \oplus (\mathbf{E}^{n_-}_{n_+}/\mathbf{D}^{n_-}_{n_+})$ to the cohomology class determined by the cocycle $M_1(z|f) + \{t(z), f(z)\}$, where $t \in \mathbf{E}^{n_-}_{n_+}$ belongs to the equivalence class T.

Theorem 3

Let $n_+ \geq 4$ or ($n_+ = 2$ and $n_- \geq 4$)

Let V_2 be the subspace of $C_2(D^{n_-}_{n_+}, D^{n_-}_{n_+})$ generated by the bilinear mappings $m_{2|1}$ and $m_{2|3}$ from $(D^{n_-}_{n_+})^2$ to $D^{n_-}_{n_+}$, which are defined by the relations

$$m_{2|1}(z|f,g) = f(z)\left(\frac{\overleftarrow{\partial}}{\partial z^A}\omega^{AB}\frac{\partial}{\partial z^B}\right)^3 g(z), \tag{6}$$

$$m_{2|3}(z|f,g) = \bar{g}\mathcal{E}_z f - (-1)^{\varepsilon(f)\varepsilon(g)}\bar{f}\mathcal{E}_z g. \tag{7}$$

Then the cochains $m_{2|1}$ and $m_{2|3}$ are independent nontrivial cocycles, dim$V_2 = 2$, and there is a natural isomorphism $V_2 \oplus (E^{n_-}_{n_+}/D^{n_-}_{n_+}) \simeq H^2_{\mathrm{ad}}$ taking $(M_2, T) \in V_2 \oplus (E^{n_-}_{n_+}/D^{n_-}_{n_+})$ to the cohomology class determined by the cocycle

$$M_2(z|f,g) - \{t(z), f(z)\}\bar{g} + (-1)^{\varepsilon(f)\varepsilon(g)}\{t(z), g(z)\}\bar{f},$$

where $t \in E^{n_-}_{n_+}$ belongs to the equivalence class T.

Theorem 4

1. *Let $n_+ = 2$, $0 \leq n_- \leq 3$. Let $N_1(f) = -2\Lambda(x_2)\int du\theta(x_1 - y_1)f(u)$, where $\Lambda \in C^\infty(\mathbb{R})$ be a function such that $\frac{d}{dx}\Lambda \in \mathcal{D}(\mathbb{R})$ and $\Lambda(-\infty) = 0,\ \Lambda(+\infty) = 1$. Let*

$$\begin{aligned}
z &= (x_1, x_2, \xi_1, \ldots, \xi_{n_-}), \quad u = (y_1, y_2, \eta_1, \ldots, \eta_{n_-}) \\
\Theta(z|f) &= \int du\delta(x_1 - y_1)\theta(x_2 - y_2)f(u), \\
N^E_2(z|f,g) &= \Theta(z|\partial_2 fg) - \Theta(z|f\partial_2 g) - 2(-1)^{n_- \varepsilon(f)}\partial_2 f(z)\Theta(z|g) + 2\Theta(z|f)\partial_2 g(z), \\
N^D_2(f,g) &= N^E_2(f,g) + d^{\mathrm{ad}}_1 N_1(f,g), && (8) \\
\Delta(z|f) &= \int du\delta(x-y)f(u), && (9) \\
\Delta_\alpha(z|f) &= \int du\eta_\alpha\delta(x-y)f(u). && (10)
\end{aligned}$$

Then the bilinear mappings $L^{n_-}_2$ from $(D^{n_-}_2)^2$ to $E^{n_-}_2$ defined by the relations

$$\begin{aligned}
L^0_2(x|f,g) &= N^D_2(x|f,g) + \frac{1}{2}\left(x^i\partial_i f(x)\right)g(x) - \frac{1}{2}f(x)\left(x^i\partial_i g(x)\right), \\
L^1_2(z|f,g) &= N^D_2(z|f,g) - \Delta(z|f)g(z) + (-1)^{\varepsilon(f)}f(z)\Delta(z|g) \\
&\quad -\frac{2}{3}(-1)^{\varepsilon(f)}\left(\xi^1\partial_{\xi^1}f(z)\right)\Delta(z|g), && (11) \\
L^2_2(z|f,g) &= N^D_2(z|f,g) - \Delta(z|f)g(z) + f(z)\Delta(z|g) \\
L^3_2(z|f,g) &= N^D_2(z|f,g) - \Delta(z|f)g(z) + (-1)^{\varepsilon(f)}f(z)\Delta(z|g) + \\
&\quad +\partial_{\xi^\alpha}f(z)\Delta_\alpha(z|g) - (-1)^{\varepsilon(f)}\Delta_\alpha(z|f)\partial_{\xi^\alpha}g(z),
\end{aligned}$$

are cocycles and maps $(D_2^{n-})^2$ *to* D_2^{n-}.

2. *Let* $n_+ = 2$ *and* $0 \le n_- \le 3$. *Let* V_2^{n-} *be the subspace of* $C_2(D_2^{n-}, D_2^{n-})$ *generated by the cocycles* M_2^1, M_2^3 *and* L_2^{n-}, *where the cocycles* M_2^1 *and* M_2^3 *are defined in Theorem 3.*

   Then there is a natural isomorphism $V_2^{n-} \oplus (E_2^{n-}/D_2^{n-}) \simeq H^2_{\mathrm{ad}}$ *taking* $(M_2, T) \in V_2^{n-} \oplus (E_2^{n-}/D_2^{n-})$ *to the cohomology class determined by the cocycle*

$$M_2(z|f,g) - \{t(z), f(z)\}\bar{g} + (-1)^{\varepsilon(f)\varepsilon(g)}\{t(z), g(z)\}\bar{f},$$

   where $t \in E_2^{n-}$ *belongs to the equivalence class* T.

1.3 Central extensions

Let $\mathbf{L}$ be the central extension of the superalgebra L, i.e. $\mathbf{L} = L \oplus c$, $\mathbf{f} = f + a\mathfrak{I} \in \mathbf{L}$, $a\mathfrak{I} \in c$, $\varepsilon(\mathfrak{I}) = 0$, $a \in \mathbb{K}$. The bracket in $\mathbf{L}$ we denote as $[\mathbf{f}, \mathbf{g}] = -(-1)^{\varepsilon(\mathbf{f})\varepsilon(\mathbf{g})}[\mathbf{g}, \mathbf{f}]$,

$$[f, g] = \{f, g\} + C(f, g)\mathfrak{I},\ C(f, g) \in \mathbb{K},\ [\mathbf{f}, \mathfrak{I}] = 0,$$

where $C(f, g) = -(-1)^{\varepsilon(f)\varepsilon(g)}C(g, f)$, $\varepsilon(C) = 0$, is a generator of the second cohomology in the trivial representation of the algebra L:

$$d_2^{\mathrm{tr}}C(f, g, h) = C(\{f, g\}, h) - (-1)^{\varepsilon(g)\varepsilon(h)}C(\{f, h\}, g) - C(f, \{g, h\}) = 0.$$

It follows from Theorem 1, that the Poisson superalgebra $D_{n_+}^{n_-}$ has the central extension $\mathbf{D}_{n_+}^{n_-}$ either if n_- is odd or if $n_- = n_+ + 4$.

Consider $C_p(\mathbf{L}, \mathbf{L})$.

If $\mathbf{M}_p \in C_p(\mathbf{L}, \mathbf{L})$ then $\mathbf{M}_p(\mathbf{f}, ...) = M_p(\mathbf{f}, ...) + m_p(\mathbf{f}, ...)\mathfrak{I} \in \mathbf{L}$, where $M_p \in C_p(\mathbf{L}, L)$, $m_p \in C_p(\mathbf{L}, \mathbb{K})$, $\varepsilon(\mathbf{M}_p) = \varepsilon(M_p) = \varepsilon(m_p) = \varepsilon_{M_p}$.

The differential $\mathbf{d}^{\mathrm{ad}}$ is defined in a usual way.

For linear forms, the differential has the form:

$$\mathbf{d}_1^{\mathrm{ad}}\mathbf{M}_1(\mathbf{f}, \mathbf{g}) = [\mathbf{M}_1(\mathbf{f}), \mathbf{g}] - (-1)^{\varepsilon(\mathbf{f})\varepsilon(\mathbf{g})}[\mathbf{M}_1(\mathbf{g}), \mathbf{f}] - \mathbf{M}_1([\mathbf{f}, \mathbf{g}])$$

and can be expressed on the decomposition of $\mathbf{L}$ as

$$\mathbf{d}_1^{\mathrm{ad}}\mathbf{M}_1(\mathfrak{I}, \mathfrak{I}) \equiv 0 \tag{12}$$

$$\mathbf{d}_1^{\mathrm{ad}}\mathbf{M}_1(f, \mathfrak{I}) = -\{M_1(\mathfrak{I}), f\} - C(M_1(\mathfrak{I}), f)\mathfrak{I}, \tag{13}$$

$$\mathbf{d}_1^{\mathrm{ad}}\mathbf{M}_1(f, g) = d_1^{\mathrm{ad}}M_1(f, g) - M_1(\mathfrak{I})C(f, g) + \gamma(f, g)\mathfrak{I}, \tag{14}$$

where

$$\gamma(f, g) = C(M_1(f), g) - (-1)^{\varepsilon(f)\varepsilon(g)}C(M_1(g), f) - m_1(\{f, g\}) - m_1(\mathfrak{I})C(f, g). \tag{15}$$

For bilinear forms, the differential has the form:

$$\begin{aligned}
\mathbf{d}_2^{\mathrm{ad}}\mathbf{M}_2(\mathbf{f}, \mathbf{g}, \mathbf{h}) \;\; = \;\; & (-1)^{\varepsilon(\mathbf{g})\varepsilon(\mathbf{h})}[\mathbf{M}_2(\mathbf{f}, \mathbf{h}), \mathbf{g}] - [\mathbf{M}_2(\mathbf{f}, \mathbf{g}), \mathbf{h}] - \\
& -(-1)^{\varepsilon(\mathbf{f})\varepsilon(\mathbf{g}) + \varepsilon(\mathbf{f})\varepsilon(\mathbf{h})}[\mathbf{M}_2(\mathbf{g}, \mathbf{h}), \mathbf{f}] - \\
& -\mathbf{M}_2([\mathbf{f}, \mathbf{g}], \mathbf{h}) + (-1)^{\varepsilon(\mathbf{g})\varepsilon(\mathbf{h})}\mathbf{M}_2([\mathbf{f}, \mathbf{h}], \mathbf{g}) + \mathbf{M}_2(\mathbf{f}, [\mathbf{g}, \mathbf{h}])
\end{aligned}$$

and can be expressed on the decomposition of $\mathbf{L}$ as

$$\begin{aligned}
&\mathbf{d}_2^{\mathrm{ad}}\mathbf{M}_2(f,g,h) = d_2^{\mathrm{ad}}M_2(f,g,h) + \\
&\quad +[M_2(f,\mathcal{I})C(g,h) - (-1)^{\varepsilon(f)\varepsilon(g)}M_2(g,\mathcal{I})C(f,h) + (-1)^{\varepsilon(f)\varepsilon(h)+\varepsilon(g)\varepsilon(h)}M_2(h,\mathcal{I})C(f,g)] + \\
&\quad +[(-1)^{\varepsilon(g)\varepsilon(h)}C(M_2(f,h),g) - C(M_2(f,g),h) - (-1)^{\varepsilon(f)\varepsilon(g)+\varepsilon(f)\varepsilon(h)}C(M_2(g,h),f) + \\
&\quad +m_2(f,\mathcal{I})C(g,h) - (-1)^{\varepsilon(f)\varepsilon(g)}m_2(g,\mathcal{I})C(f,h) + (-1)^{\varepsilon(f)\varepsilon(h)+\varepsilon(g)\varepsilon(h)}m_2(h,\mathcal{I})C(f,g) - \\
&\quad -d_2^{\mathrm{tr}}m_2(f,g,h)]\mathcal{I}, \qquad (16)
\end{aligned}$$

$$\begin{aligned}
&\mathbf{d}_2^{\mathrm{ad}}\mathbf{M}_2(f,g,\mathcal{I}) = d_1^{\mathrm{ad}}\tilde{M}_1(f,g) + [C(M_2(f,\mathcal{I}),g) - (-1)^{\varepsilon(f)\varepsilon(g)}C(M_2(g,\mathcal{I}),f) - \\
&\quad -m_2(\{f,g\},\mathcal{I})]\mathcal{I}, \qquad (17) \\
&\mathbf{d}_2^{\mathrm{ad}}\mathbf{M}_2(f,\mathcal{I},\mathcal{I}) = \mathbf{d}_2^{\mathrm{ad}}\mathbf{M}_2(\mathcal{I},\mathcal{I},\mathcal{I}) \equiv 0, \qquad (18)
\end{aligned}$$

where

$$\tilde{M}_1(f) = M_2(f,\mathcal{I}).$$

1.4 Cohomologies of central extension of the Poisson superalgebra

The following theorems are proved in the next sections.

Theorem 5

Let $n_- = 2k+1$. Let the bilinear mappings $\mathbf{M}_2^1$ and $\mathbf{M}_2^3$ from $(\mathbf{D}_{n_+}^{n_-})^2$ to $\mathbf{D}_{n_+}^{n_-}$ are defined by the relations

$$\begin{aligned}
\mathbf{M}_2^1(z|f,g) &= m_{2|1}(z|f,g),\ \mathbf{M}_2^1(f,\mathcal{I}) = 0, \qquad (19) \\
\mathbf{M}_2^3(z|f,g) &= m_{2|3}(z|f,g),\ \mathbf{M}_2^3(f,\mathcal{I}) = 0, \qquad (20)
\end{aligned}$$

and the bilinear mapping $\mathbf{L}$ from $(\mathbf{D}_2^1)^2$ to $\mathbf{D}_2^1$ is defined by the relations

$$\mathbf{L}(z|f,g) = L_2^1(z|f,g) + \frac{20}{3}\left(\int dz\xi f(z)\bar{g} - \int dz\xi g(z)\bar{f}\right)\mathcal{I},\ \mathbf{L}(f,\mathcal{I}) = 0, \qquad (21)$$

where $m_{2|1}(z|f,g)$, $m_{2|1}(z|f,g)$, and $L_2^1(z|f,g)$ are defined by (6), (7), and (11) respectively.

Let V_2 be the subspace of $C_2(\mathbf{D}_{n_+}^{n_-},\mathbf{D}_{n_+}^{n_-})$ generated by the bilinear mappings $\mathbf{M}_2^1$, $\mathbf{M}_2^3$, and $\delta_{k,0}\delta_{2,n_+}\mathbf{L}$.

Then $dimV_2 = 2+\delta_{k,0}\delta_{2,n_+}$, and there is a natural isomorphism $V_2\oplus(E_{n_+}^{2k+1}/D_{n_+}^{2k+1}) \simeq H_{\mathrm{ad}}^2$ taking $(\mathbf{M}_2,T) \in V_2\oplus(E_{n_+}^{n_-}/D_{n_+}^{n_-})$ to the cohomology class determined by the cocycle

$$\mathbf{M}_2(z|\mathbf{f},\mathbf{g}) - \{t(z),f(z)\}\bar{g} + (-1)^{\varepsilon(f)\varepsilon(g)}\{t(z),g(z)\}\bar{f},$$

where $t \in E_{n_+}^{n_-}$ belongs to the equivalence class T.

Theorem 6

Let $n_- = n_+ + 4$. Let the bilinear mappings $\mathbf{M}_2^3$ and $\mathbf{Q}_\tau$ from $(\mathbf{D}_{n_+}^{n_-})^2$ to $\mathbf{D}_{n_+}^{n_-}$ are defined by the relations

$$\begin{aligned}
&\mathbf{M}_2^3(z|f,g) = \bar{g}\mathcal{E}_z f - (-1)^{\varepsilon(f)\varepsilon(g)}\bar{f}\mathcal{E}_z g, \quad \mathbf{M}_2^3(\mathbf{f},\mathcal{I}) = 0, \qquad (22) \\
&\mathbf{Q}_\tau(z|f,g) = \{\tau(z),g(z)\}\bar{f} - (-1)^{\varepsilon(f)\varepsilon(g)}\{\tau(z),f(z)\}\bar{g} + \left(C_2^2(\tau,g)\bar{f} - (-1)^{\varepsilon(f)\varepsilon(g)}C_2^2(\tau,f)\bar{g}\right)\mathcal{I} \\
&\qquad \mathbf{Q}_\tau(z|\mathbf{f},\mathcal{I}) = 0, \qquad (23)
\end{aligned}$$

where C_2^2 *is defined by (5) and* $\zeta \in E_{n_+}^{n_-}$.

Let V_2 *be the subspace of* $C_2(\mathbf{D}_{n_+}^{n_-}, \mathbf{D}_{n_+}^{n_-})$ *generated by the bilinear mapping* $\mathbf{M}_2^3$.

Then $dim V_2 = 1$ *and there is a natural isomorphism* $V_2 \oplus (E_{n_+}^{2k+1}/D_{n_+}^{2k+1}) \simeq H_{\rm ad}^2$ *taking* $(\mathbf{M}_2, T) \in V_2 \oplus (E_{n_+}^{n_-}/D_{n_+}^{n_-})$ *to the cohomology class determined by the cocycle*

$$\mathbf{M}_2(z|\mathbf{f}, \mathbf{g}) + \mathbf{Q}_\tau(z|\mathbf{f}, \mathbf{g})$$

where $\tau \in E_{n_+}^{n_-}$ *belongs to the equivalence class* T.

1.5 Deformations of the Lie superalgebra

Let L be a topological Lie superalgebra over $\mathbb{K}$ with Lie superbracket $\{\cdot,\cdot\}$, $\mathbb{K}[[\hbar^2]]$ be the ring of formal power series in $\hbar^2$ over $\mathbb{K}$, and $L[[\hbar^2]]$ be the $\mathbb{K}[[\hbar^2]]$-module of formal power series in $\hbar^2$ with coefficients in L. We endow both $\mathbb{K}[[\hbar^2]]$ and $L[[\hbar^2]]$ by the direct-product topology. The grading of L naturally determines a grading of $L[[\hbar^2]]$: an element $f = f_0 + \hbar^2 f_1 + \ldots$ has a definite parity $\varepsilon(f)$ if $\varepsilon(f) = \varepsilon(f_j)$ for all $j = 0, 1, ...$ Every p-linear separately continuous mapping from L^p to L (in particular, the bracket $\{\cdot,\cdot\}$) is uniquely extended by $\mathbb{K}[[\hbar^2]]$-linearity to a p-linear separately continuous mapping over $\mathbb{K}[[\hbar^2]]$ from $L[[\hbar^2]]^p$ to $L[[\hbar^2]]$. A (continuous) formal deformation of L is by definition a $\mathbb{K}[[\hbar^2]]$-bilinear separately continuous Lie superbracket $C(\cdot,\cdot)$ on $L[[\hbar^2]]$ such that $C(f,g) = \{f,g\} \mod \hbar^2$ for any $f, g \in L[[\hbar^2]]$. Obviously, every formal deformation C is expressible in the form

$$C(f,g) = \{f,g\} + \hbar^2 C_1(f,g) + \hbar^4 C_2(f,g) + \ldots, \quad f, g \in L, \tag{24}$$

where C_j are separately continuous skew-symmetric bilinear mappings from $L \times L$ to L (2-cochains with coefficients in the adjoint representation of L). Formal deformations C^1 and C^2 are called equivalent if there is a continuous $\mathbb{K}[[\hbar^2]]$-linear operator $T : L[[\hbar^2]] \to L[[\hbar^2]]$ such that $TC^1(f,g) = C^2(Tf, Tg)$, $f, g \in L[[\hbar^2]]$.

Here as well as in [3], we use more restricting definition

Definition 7 *Formal deformations* C^1 *and* C^2 *are called similar if there are a continuous* $\mathbb{K}[[\hbar^2]]$*-linear operators* T, T_1: $L[[\hbar^2]] \to L[[\hbar^2]]$ *such that* $TC^1(f,g) = C^2(Tf, Tg)$, $f, g \in L[[\hbar^2]]$ *and* $Tf = f + \hbar^2 T_1 f$.

1.6 Deformations of central extension of the Poisson superalgebra

The following theorems are proved in subsequent sections:

Theorem 8 *Let* $n_- = 2k+1$. *Let* $\mathcal{M}_\kappa(z|f,g) = \frac{1}{\hbar\kappa} f(z) \sinh\left(\hbar\kappa \frac{\overleftarrow{\partial}}{\partial z^A} \omega^{AB} \frac{\partial}{\partial z^B}\right) g(z)$. *For given* κ, *let* $\zeta(z), w(z) \in E_{n_+}^{2k+1}[[\hbar^2]]$ *satisfy the following conditions*

$$i) \qquad \varepsilon(\zeta) = 1, \quad \varepsilon(w) = 0; \tag{25}$$

$$ii) \qquad \mathcal{M}_\kappa(z|\zeta,\zeta) + w(z) \in D_{n_+}^{2k+1}[[\hbar^2]]. \tag{26}$$

Let $\mathbf{N}_{2|\kappa,\zeta,w}(\mathbf{f},\mathbf{g}) = N_{2|\kappa,\zeta,w}(z|\mathbf{f},\mathbf{g}) + n_{2|\kappa,\zeta,w}(\mathbf{f},\mathbf{g})\mathfrak{I}$, *where*

$$\begin{aligned}
&N_{2|\kappa,\zeta,w}(z|f,g) = \mathcal{M}_\kappa(z|f+\hbar^2\zeta\bar{f}, g+\hbar^2\zeta\bar{g}) + \hbar^4 w(z)\bar{f}\bar{g},\\
&N_{2|\kappa,\zeta,w}(z|f,\mathfrak{I}) = \mathcal{M}_\kappa(z|w, f+\hbar^2\zeta\bar{f}),\\
&n_{2|\kappa,\zeta,w}(f,g) = \bar{f}\bar{g},\\
&n_{2|\kappa,\zeta,w}(f,\mathfrak{I}) = 0,\\
&\mathbf{N}_{2|\kappa,\zeta,w}(z|\mathfrak{I},\mathfrak{I}) = 0.
\end{aligned}$$

Then

1. *every continuous formal deformation of the superalgebra* $\mathbf{D}_{n_+}^{2k+1}$ *is similar to the superbracket*
$$[\mathbf{f},\mathbf{g}]_{\kappa,\zeta,w} = \mathbf{N}_{2|\kappa,\zeta,w}(\mathbf{f},\mathbf{g})$$
with some κ *and* $\zeta(z), w(z) \in E_{n_+}^{2k+1}[[\hbar^2]]$ *satisfying (25)-(26) ;*

2. *the superbracket* $[\mathbf{f},\mathbf{g}]_{\kappa,\zeta_1,w_1}$ *is similar to the superbracket* $[\mathbf{f},\mathbf{g}]_{\kappa,\zeta,w}$, *if* $\zeta_1(z)-\zeta(z) \in D_{n_+}^{n_-}[[\hbar^2]]$ *and* $w_1(z)-w(z) \in D_{n_+}^{n_-}[[\hbar^2]]$.

Theorem 9 *Let* $n_- = n_+ + 4$. *Let* $\zeta(z) \in E_{n_+}^{n_++4}[[\hbar^2]]$, $c_3 \in \mathbb{K}[[\hbar^2]]$, $C(f,g) = \int dz\left(f(z)\mathcal{E}_z g(z) - (-1)^{\varepsilon(f)\varepsilon(g)}g(z)\mathcal{E}_z f(z)\right)$ *and*

$$\begin{aligned}
&\mathbf{S}_{2|\zeta,c_3}(\mathbf{f},\mathbf{g}) = S_{2|\zeta,c_3}(z|\mathbf{f},\mathbf{g}) + s_{2|\zeta,c_3}(\mathbf{f},\mathbf{g})\mathfrak{I},\\
&S_{2|\zeta,c_3}(z|f,g) = \{f(z),g(z)\} + c_3\left(\bar{f}\mathcal{E}_z g(z) - (-1)^{\varepsilon(f)\varepsilon(g)}\bar{g}\mathcal{E}_z f(z)\right) + \\
&\qquad\qquad + \hbar^2\left(\{\zeta(z),g(z)\}\bar{f} - (-1)^{\varepsilon(f)\varepsilon(g)}\{\zeta(z),f(z)\}\bar{g}\right),\\
&s_{2|\zeta,c_3}(f,g) = C(f,g) + C(\hbar^2\zeta,g)\bar{f} - (-1)^{\varepsilon(f)\varepsilon(g)}C(\hbar^2\zeta,f)\bar{g},\\
&S_{2|\zeta,c_3}(\mathbf{f},\mathfrak{I}) = s_{2|\zeta,c_3}(\mathbf{f},\mathfrak{I}) = 0.
\end{aligned}$$

Then

1. *every continuous formal deformation of the superalgebra* $\mathbf{D}_{n_+}^{n_++4}$ *is similar to the superbracket*
$$[\mathbf{f},\mathbf{g}]_{\zeta,c_3} = \mathbf{S}_{2|\zeta,c_3}(\mathbf{f},\mathbf{g})$$
with some c_3 *and* $\zeta(z) \in E_{n_+}^{2k+1}[[\hbar^2]]$ *;*

2. *the superbracket* $[\mathbf{f},\mathbf{g}]_{\zeta_1,c_3}$ *is similar to the superbracket* $[\mathbf{f},\mathbf{g}]_{\zeta,c_3}$, *if* $\zeta_1(z)-\zeta(z) \in D_{n_+}^{n_-}[[\hbar^2]]$.

2 Superalgebra $\mathbf{D}_{n_+}^{2k+1}$

2.1 Second adjoint cohomology

In the superalgebra under consideration we have $C(f,g) = \bar{f}\bar{g}$ and $[f,g] = \{f,g\} + \bar{f}\bar{g}\mathfrak{I}$.

Consider the cohomology equation $\mathbf{d}_2^{\mathrm{ad}}\mathbf{M}_2(\mathbf{f},\mathbf{g},\mathbf{h}) = 0$. It follows from (17) that $d_1^{\mathrm{ad}}\tilde{M}_1(z|f,g) = 0$ and so, due to Theorem 2

$$M_2(z|f,\mathfrak{I}) = t^0\mathcal{E}_z f(z) + \{t'(z), f(z)\}\,,\quad t'(z) \in E_{n_+}^{2k+1},\quad \varepsilon(t^0) = \varepsilon(t'(z)) = \varepsilon_{M_2}, \tag{27}$$

It follows from (17) also that $m_2(\{f,g\},\mathfrak{I}) = (2+n_+-n_-)t^0\bar{f}\bar{g}$ which implies $t^0 = 0$ and $m_2(\{f,g\},\mathfrak{I}) = 0$. In its turn, this gives $m_2(f,\mathfrak{I}) = m_2\bar{f}$, $m_2 = \text{const}$, $\varepsilon(m_2) = \varepsilon_{M_2}+1$.

So, we have proved

Proposition 10

$$\begin{aligned}
&M_2(z|f,\mathfrak{I}) = \{t'(z), f(z)\}\,,\ m_2(f,\mathfrak{I}) = m_2\bar{f},\\
&t'(z)\in E^{2k+1}_{n_+},\ \varepsilon(t'(z)) = \varepsilon_{M_2},\ \varepsilon(m_2) = \varepsilon_{M_2}+1.
\end{aligned}$$

Further, it follows from (16) that $d_2^{\mathrm{ad}}M_2'(z|f,g,h) = 0$, where $M_2'(z|f,g) = M_2(z|f,g) - \bar{f}\bar{g}t'(z)$.

So, the proposition follows from Theorem 3 and the condition $M_2(z|f,g)\in D^{2k+1}_{n_+}$

Proposition 11

$$\begin{aligned}
M_2(z|f,g) &= c_1m_{2|1}(z|f,g) + t_D(z)m_{2|2}(z|f,g) + c_3m_{2|3}(z|f,g) + m_{2|\zeta}(z|f,g) + \\
&+ c_5^k m_{2|5}(z|f,g) + d_1^{\mathrm{ad}}\varphi_D(z|f,g),
\end{aligned}$$

where

$$\begin{aligned}
&m_{2|1}(z|f,g) = f(z)\left(\overleftarrow{\partial}_A\omega^{AB}\partial_B\right)^3 g(z),\ \bar{m}_{2|1}(|f,g) = 0,\quad \varepsilon_{m_{2|1}} = 0,\\
&m_{2|2}(z|f,g) = \bar{f}\bar{g},\\
&m_{2|3}(z|f,g) = \bar{f}\mathcal{E}_z g(z) - (-1)^{\varepsilon(f)\varepsilon(g)}\bar{g}\mathcal{E}_z f(z),\ \bar{m}_{2|3}(|f,g) = 0,\quad \varepsilon_{m_{2|3}} = n_-,\\
&t_D(z) = t'(z) + c_2 \in D^{2k+1}_{n_+},\ \varepsilon(t_D(z)) = \varepsilon_{M_2},\\
&m_{2|\zeta}(z|f,g) = \{\zeta(z), g(z)\}\bar{f} - (-1)^{\varepsilon(f)\varepsilon(g)}\{\zeta(z), f(z)\}\bar{g},\ \bar{m}_{2|\zeta}(|f,g) = 0,\\
&\qquad \zeta(z)\in E^{2k+1}_{n_+}/D^{2k+1}_{n_+},\ \varepsilon(\zeta(z)) = \varepsilon_{M_2}+n_-,\\
&m_{2|5}(z|f,g) = \delta_{n_+,2}L_2^{2k+1},\quad \varepsilon_{m_{2|5}} = 1,\quad L_2^{2k+1} = 0 \textit{ for } k>1,\\
&\varphi_D(z|f)\in C_1(D^{2k+1}_{n_+}, D^{2k+1}_{n_+}),\quad \varepsilon_{\varphi_D} = \varepsilon_{M_2},
\end{aligned}$$

$$\begin{aligned}
&\bar{M}_2(|f,g) = \bar{t}_D\bar{f}\bar{g} + c_5^k a_k\omega(f,g) - \bar{\zeta}(|\,\{f,g\}),\\
&\omega(f,g) = \delta_{n_+,2}\int dxd\eta d\xi f(x,\xi)g(x,\eta),\ a_k = \left\{\begin{array}{l}\frac{20}{3},\ k=0,\\ 6,\ k=1,\\ 0,\ k\geq 2,\end{array}\right.
\end{aligned}$$

and the bilinear forms L_2^i are defined by (11).

To find $m_2(f,g)$ and the relations between the parameters in Proposition 11 , consider Eq. (16). It gives

$$d_2^{\mathrm{tr}}m_2'(f,g,h) = 3(m_2-\bar{t}_D)\bar{f}\bar{g}\bar{h} - c_5^k a_{n_-}[\omega(f,g)\bar{h} + \omega(f,h)\bar{g} + \omega(g,h)\bar{f}], \qquad (28)$$

where

$$m_2'(f,g) = m_2(f,g) - [\bar{\varphi}_D(|f)\bar{g} - (-1)^{\varepsilon(f)\varepsilon(g)}\bar{\varphi}_D(|g)\bar{f}].$$

Evidently, there exist such functions $f, g, h \in D^{2k+1}_{n_+}$ that $\bar{f}\bar{g}\bar{h} \neq 0$, $\{f, g\} = \{g, h\} = \{h, f\} = 0$ and $\omega(f, g) = \omega(g, h) = \omega(h, f) = 0$. Indeed, let

$$\mathrm{supp}\,(f) \bigcap \mathrm{supp}\,(g) = \mathrm{supp}\,(g) \bigcap \mathrm{supp}\,(h) = \mathrm{supp}\,(h) \bigcap \mathrm{supp}\,(f) = \varnothing.$$

So, Eq. (28) implies

Proposition 12 $m_2 = \bar{t}_D$,

and as a consequence

$$d_2^{\mathrm{tr}} m_2'(f, g, h) = -c_5^k a_k[\omega(f, g)\bar{h} + \omega(f, h)\bar{g} + \omega(g, h)\bar{f}] \tag{29}$$

Proposition 13 *$c_5^k = 0$ for $k \geq 1$, i.e. $c_5^k = c_5\delta_{0,k}$, where $c_5 \in \mathbb{K}$*

Proof. Consider Eq. (29) for the functions $f = g = h = \varphi(x)\delta(\xi)$ ■

Proposition 14 *Let $k = 0$. Then*

$$\omega(f, g)\bar{h} + \omega(f, h)\bar{g} + \omega(g, h)\bar{f} = \frac{3}{20} d_2^{\mathrm{tr}} \chi_2(f, g, h), \tag{30}$$

where

$$\chi_2(f, g) = \frac{20}{3}\delta_{n_+,2}\left(\int dz\xi f(z)\bar{g} - \int dz\xi g(z)\bar{f}\right). \tag{31}$$

Proof.

$$\omega(f, g)\bar{h} + \omega(f, h)\bar{g} + \omega(g, h)\bar{f} = \delta_{n_+,2}\Big(\int dx f_{n_-}(x) g_{n_-}(x) \int dy h_{n_-}(y) +$$
$$+ \int dx f_{n_-}(x) h_{n_-}(x) \int dy g_{n_-}(y) + \int dx g_{n_-}(x) h_{n_-}(x) \int dy f_{n_-}(y)\Big) =$$
$$= \delta_{n_+,2}\left(\int dz\xi \{f(z), g(z)\}\bar{h} + \mathrm{cycle}(f, g, h)\right)$$

■

Thus, we obtain

$$M_2(z|f, g) = c_1 m_{2|1}(z|f, g) + c_3 m_{2|3}(z|f, g) + m_{2|\zeta}(z|f, g) + c_5\delta_{k,0}\delta_{2,n_+} L_2^1(z|f, g) +$$
$$+ t_D(z) m_{2|2}(z|f, g) + d_1^{\mathrm{ad}}\varphi(z|f, g),$$
$$m_2(f, g) = \bar{\varphi}_D(|f)\bar{g} - (-1)^{\varepsilon(f)\varepsilon(g)}\bar{\varphi}_D(|g)\bar{f} + c_5\delta_{k,0}\chi_2(f, g) + b_2\bar{f}\bar{g} + \mu_1(\{f, g\}),$$
$$M_2(z|f, \mathfrak{I}) = \{t_D(z), f(z)\},\ m_2(f, \mathfrak{I}) = \bar{t}\bar{f}.$$

where $\mu_1 \in C_1(D^{2k+1}_{n_+}, \mathbb{K})$, $t_D \in D^{2k+1}_{n_+}$, $\varphi_D \in C_1(D^{2k+1}_{n_+}, D^{2k+1}_{n_+})$.

Let $\mathbf{M}_{1D}(\mathbf{f}) = \varphi_D(z|f) - \mu_1(f)\mathfrak{I}$, $\mathbf{M}_{1D}(\mathfrak{I}) = -t_D(z) - b_2\mathfrak{I}$. Then

$$\mathbf{M}_2(\mathbf{f}, \mathbf{g}) = \mathbf{M}_{2|1}(\mathbf{f}, \mathbf{g}) + \mathbf{d}_1^{\mathrm{ad}}\mathbf{M}_{1D}(\mathbf{f}, \mathbf{g}) \tag{32}$$

where

$$\mathbf{M}_{2|1}(f,g)=c_1m_{2|1}(z|f,g)+c_3m_{2|3}(z|f,g)+m_{2|\zeta}(z|f,g)+c_5\delta_{k,0}\delta_{2,n_+}L_2^1(z|f,g)+$$
$$+c_5\delta_{k,0}\chi_2(f,g)\mathfrak{I},$$
$$\mathbf{M}_{2|1}(f,\mathfrak{I})=0$$

Finally, we have, that up to coboundary, the cocycle $\mathbf{M}_2(\mathbf{f},\mathbf{g})$ has the form

$$M_2(z|f,g)=c_1m_{2|1}(z|f,g)+c_3m_{2|3}(z|f,g)+m_{2|\zeta}(z|f,g)+c_5\delta_{k,0}\delta_{2,n_+}L_2^1(z|f,g), \quad (33)$$
$$m_2(f,g)=\delta_{k,0}c_5\chi_2(f,g), \quad (34)$$
$$\mathbf{M}_2(f,\mathfrak{I})=0. \quad (35)$$

2.1.1 Independence and non-triviality

Suppose that

$$\mathbf{M}_2(\mathbf{f},\mathbf{g})=\mathbf{d}_1^{\mathrm{ad}}\mathbf{M}_1(\mathbf{f},\mathbf{g}).$$

This relation yields

$$c_1m_{2|1}(z|f,g)+c_3m_{2|3}(z|f,g)+c_5\delta_{k,0}\delta_{2,n_+}L_2^1(z|f,g)=d_1^{\mathrm{ad}}M_1(z|f,g)-\bar{f}\bar{g}M_1(z|\mathfrak{I}). \quad (36)$$

Let

$$z\bigcap\left[\mathrm{supp}(f)\bigcup\mathrm{supp}(g)\right]=\mathrm{supp}(f)\bigcap\mathrm{supp}(g)=\varnothing.$$

It follows from (36)

$$\bar{f}\bar{g}M_1(z|\mathfrak{I})=0 \implies M_1(z|\mathfrak{I})=0 \implies$$
$$c_1m_{2|1}(z|f,g)+c_3m_{2|3}(z|f,g)+c_5\delta_{k,0}\delta_{2,n_+}L_2^1(z|f,g)=d_1^{\mathrm{ad}}M_1(z|f,g).$$

This equation has solution for $c_1=c_3=c_5=0$ only.

2.2 Deformations of Lie superalgebra $\mathbf{D}_{n_+}^{2k+1}$

In this section, we find the general form of the deformation of Lie superalgebra $\mathbf{D}_{n_+}^{2k+1}$, $[f,g]_*$,

$$[\mathbf{f},\mathbf{g}]_*= \quad M_2^*(z|\mathbf{f},\mathbf{g})+m_2^*(\mathbf{f},\mathbf{g})\mathfrak{I}=\sum_{l=0}^{\infty}\hbar^{2l}\mathbf{M}_{2l}(\mathbf{f},\mathbf{g}),$$
$$M_2^*(z|\mathbf{f},\mathbf{g})\in D_{n_+}^{2k+1}[[\hbar^2]],\ m_2^*(\mathbf{f},\mathbf{g})\in\mathbb{K}[[\hbar^2]],\ \varepsilon_{M_2^*}=\varepsilon_{m_2^*}=0,$$

satisfying the Jacoby identity.

$$(-1)^{\varepsilon(\mathbf{f})\varepsilon(\mathbf{h})}[[\mathbf{f},\mathbf{g}]_*,\mathbf{h}]_*+(-1)^{\varepsilon(\mathbf{g})\varepsilon(\mathbf{h})}[[\mathbf{h},\mathbf{f}]_*,\mathbf{g}]_*+(-1)^{\varepsilon(\mathbf{f})\varepsilon(\mathbf{h})}[[\mathbf{g},\mathbf{h}]_*,\mathbf{f}]_*=0. \quad (37)$$

We have $M_{20}(f,g)=\{f(z),g(z)\}$, $m_{20}(f,g)=\bar{f}\bar{g}$, $M_{20}(\mathbf{f},\mathfrak{I})=m_{20}(\mathbf{f},\mathfrak{I})=0$.

For any $\kappa\in\mathbb{K}$, the Moyal-type superbracket

$$\mathcal{M}_\kappa(z|f,g)=\frac{1}{\hbar\kappa}f(z)\sinh\left(\hbar\kappa\frac{\overleftarrow{\partial}}{\partial z^A}\omega^{AB}\frac{\partial}{\partial z^B}\right)g(z)$$

is antisymmetric and satisfies the Jacobi identity.

Definition 15 *For $\zeta \in E^{2k+1}_{n_+}$, $\kappa \in \mathbb{K}$, we set*

$$\mathcal{N}_{\kappa,\zeta}(z|f,g) = \mathcal{M}_\kappa(z|f+\zeta\bar{f}, g+\zeta\bar{g}).$$

It is shown in [3], that if ζ_1 and ζ_2 belong to the same equivalence class of $E^{2k+1}_{n_+}/D^{2k+1}_{n_+}$ and $\mathcal{N}_{\kappa,\zeta_1}(z|f,g) \in D^{2k+1}_{n_+}[[\hbar^2]]$ for all $f, g \in D^{2k+1}_{n_+}[[\hbar^2]]$ then the bilinear forms $\mathcal{N}_{\kappa,\zeta_1}(z|f,g)$ and $\mathcal{N}_{\kappa,\zeta_2}(z|f,g)$ are equivalent under some similarity transformation T, mapping $D^{2k+1}_{n_+}[[\hbar^2]]$ to $D^{2k+1}_{n_+}[[\hbar^2]]$,

$$\mathcal{N}_{\kappa,\zeta_1}(z|Tf,Tg) = T\mathcal{N}_{\kappa,\zeta_2}(z|f,g). \tag{38}$$

2.2.1 $\hbar^2$-order

In $\hbar^2$-order, Eq. (37) gives an equation

$$\mathbf{d}_2^{\mathrm{ad}}\mathbf{M}_{21}(\mathbf{f},\mathbf{g},\mathbf{h}) = 0,\ \varepsilon_{\mathbf{M}_{21}} = 0,$$

general solution of which is [2]

$$\begin{aligned}
&\mathbf{M}_{21}(\mathbf{f},\mathbf{g}) = \mathbf{M}_{2|11}(\mathbf{f},\mathbf{g}) + \mathbf{d}_1^{\mathrm{ad}}\mathbf{M}_{1D1}(\mathbf{f},\mathbf{g}),\\
&M_{2|11}(z|f,g) = \frac{1}{6}\kappa_1^2 m_{2|1}(z|f,g) + m_{2|\zeta_1}(z|f,g),\\
&\varepsilon(\zeta_1(z)) = 1,\ \zeta_1(z) \in E^{2k+1}_{n_+}/D^{2k+1}_{n_+},\\
&m_{2|11}(f,g) = \mathbf{M}_{2|11}(f,\mathfrak{I}) = 0,\\
&\mathbf{M}_{1D1}(f) = \zeta_{D1}(z|f) + m_{11}(f)\mathfrak{I},\ \mathbf{M}_{1D1}(\mathfrak{I}) = t_{D1}(z) + b_1\mathfrak{I}.
\end{aligned}$$

Performing the similarity transformation $[\mathbf{f},\mathbf{g}]_* \to [\mathbf{f},\mathbf{g}]_{*T}$ with[3] $\mathbf{T}(\mathbf{f}) = \mathbf{f} - \hbar^2\mathbf{M}_{1D1}(\mathbf{f}) + O(\hbar^4)$, we can rewrite $[\mathbf{f},\mathbf{g}]_*$ in the form

$$[\mathbf{f},\mathbf{g}]_* = \mathbf{N}_{\kappa_{[1]},\zeta_{[1]}}(\mathbf{f},\mathbf{g}) + \hbar^4\mathbf{M}_{22}(\mathbf{f},\mathbf{g}) + O(\hbar^6),$$

where

$$\begin{aligned}
&\mathbf{N}_{\kappa_{[1]},\zeta_{[1]}}(f,g) = \mathcal{N}_{\kappa_{[1]},\zeta_{[1]}}(z|f,g) - \mathcal{M}_{\kappa[1]}(z|\zeta_{[1]},\zeta_{[1]})_2\bar{f}\bar{g} + \bar{f}\bar{g}\mathfrak{I},\\
&\bar{\mathcal{N}}_{\kappa_{[1]},\zeta_{[1]}}(|f,g) - \bar{\mathcal{M}}_{\kappa[1]}(|\zeta_{[1]},\zeta_{[1]})_2\bar{f}\bar{g} = O(\hbar^6),\\
&\mathbf{N}_{\kappa_{[1]},\zeta_{[1]}}(f,\mathfrak{I}) = \mathbf{N}_{\kappa_{[1]},\zeta_{[1]}}(\mathfrak{I},\mathfrak{I}) = 0,
\end{aligned}$$

$$\kappa_{[n]} = \sum_{k=1}^{n}\hbar^{2(k-1)}\kappa_k,\ \zeta_{[n]} = \sum_{k=1}^{n}\hbar^{2k}\zeta_k,$$

and $\hbar^{2(n+1)}\mathcal{M}_{\kappa[n]}(z|\zeta_{[n]},\zeta_{[n]})_{n+1}$ is the $\hbar^{2(n+1)}$-order term of expansion of $\mathcal{M}_{\kappa[n]}(z|\zeta_{[n]},\zeta_{[n]})$ in $\hbar^2$ -series.

The condition $[\mathbf{f},\mathbf{g}]_* \in \mathbf{D}^{2k+1}_{n_+}$ implies that $\mathbf{M}_{22}(\mathbf{f},\mathbf{g}) \in \mathbf{D}^{2k+1}_{n_+}$.

---

[2] We should set $c_3 = c_5 = 0$ because $\varepsilon_{m_{2|3}} = \varepsilon_{L_2^1} = 1$.

[3] Note that $\mathbf{T} = 1 + \hbar^2\mathbf{M}_1 + O(\hbar^4)$ is a similarity transformation if $M_1(z|f), M_1(z|\mathfrak{I}) \in D^{2k+1}_{n_+}[[\hbar^2]]$ for any $f \in D^{2k+1}_{n_+}[[\hbar^2]]$.

2.2.2 $\hbar^4$-order

The Jacobi identity (37) gives the following equations in $\hbar^4$-order

$$d_1^{\mathrm{ad}}\tilde{M}_{12}(z|f,g)=0,\ \tilde{M}_{12}(z|f)=M_{22}(z|f,\mathfrak{I}), \tag{39}$$

$$\bar{f}\bar{M}_{22}(|g,\mathfrak{I})-(-1)^{\varepsilon(f)\varepsilon(g)}\bar{g}\bar{M}_{22}(|f,\mathfrak{I})+m_{22}(\{f,g\},\mathfrak{I})=0. \tag{40}$$

$$\begin{aligned}&d_2^{\mathrm{ad}}M'_{22}(z|f,g,h)+[\bar{f}\bar{g}M_{22}(z|h,\mathfrak{I})-(-1)^{\varepsilon(f)\varepsilon(g)}\bar{f}\bar{h}M_{22}(z|g,\mathfrak{I})+\bar{g}\bar{h}M_{22}(z|f,\mathfrak{I})]=0,\\ &M'_{22}(z|f,g)=M_{22}(z|f,g)-\mathcal{M}_{\kappa[1]}(z|\zeta_{[1]},\zeta_{[1]})_2\bar{f}\bar{g},\end{aligned} \tag{41}$$

$$\begin{aligned}&d_2^{\mathrm{tr}}m_{22}(f,g,h)=(-1)^{\varepsilon(f)\varepsilon(h)}[(-1)^{\varepsilon(f)\varepsilon(h)}\bar{f}\bar{M}_{22}(|g,h)\\ &+(-1)^{\varepsilon(f)\varepsilon(h)}\bar{f}\bar{g}m_{22}(h,\mathfrak{I})+\mathrm{cycle}(f,g,h)].\end{aligned} \tag{42}$$

The general solution of Eq. (39) has the form

$$\begin{aligned}&M_{22}(z|f,\mathfrak{I})=t_2^0\mathcal{E}_zf(z)+\{t_2(z),f(z)\},\\ &\varepsilon(t_2^0)=\varepsilon(t_2(z))=0,\ t_2(z)\in E_{n_+}^{2k+1}.\end{aligned}$$

Further, Eq. (40) gives

$$t_2^0=0,\ m_{22}(f,\mathfrak{I})=m_{22}\bar{f},\ m_{22}=\mathrm{const},$$

such that

$$M_{22}(z|f,\mathfrak{I})=\{t_2(z),f(z)\},\ m_{22}(f,\mathfrak{I})=m_{22}\bar{f}.$$

The condition $\varepsilon_{m_{22}}=0$ gives $m_{22}=0$ and

$$m_{22}(f,\mathfrak{I})=0.$$

Further, we find from Eq. (41)

$$\begin{aligned}&M_{22}(f,g)=\frac{\kappa_1\kappa_2}{3}m_{2|1}(z|f,g)+m_{2|\zeta_2}(z|f,g)+\\ &+\left(\mathcal{M}_{\kappa[1]}(z|\zeta_{[1]},\zeta_{[1]})_2+t_2(z)+c_{22}\right)\bar{f}\bar{g}+d_1^{\mathrm{ad}}\zeta_{D2}(z|f,g),\\ &\mathcal{M}_{\kappa[1]}(z|\zeta_{[1]},\zeta_{[1]})_2+t_2(z)+c_{22}\in D_{n_+}^{2k+1},\ \zeta_{D2}(z|f)\in D_{n_+}^{2k+1},\ \zeta_2(z)\in E_{n_+}^{2k+1}/D_{n_+}^{2k+1}.\end{aligned}$$

Finally, we obtain from Eq. (42)

$$m_{22}(f,g)=\bar{\zeta}_{D2}(|f)\bar{g}-(-1)^{\varepsilon(f)\varepsilon(g)}\bar{\zeta}_{D2}(|g)\bar{f}-b_{22}\bar{f}\bar{g}-\mu_{12}(\{f,g\}).$$

Introduce notation

$$t_2(z)+c_{22}=t_{D2}(z)+w_2(z),\ t_{D2}(z)\in D_{n_+}^{2k+1},\ w_2(z)\in E_{n_+}^{2k+1}/D_{n_+}^{2k+1}.$$

Then, we can write

$$\mathbf{M}_{22}(\mathbf{f},\mathbf{g})=\mathbf{M}_{22|\mathrm{co}}(\mathbf{f},\mathbf{g})+\mathbf{d}_1^{\mathrm{ad}}\mathbf{M}_{1D2}(\mathbf{f},\mathbf{g}),$$

$$\begin{aligned}&M_{22|\mathrm{co}}(f,g)=\frac{\kappa_1\kappa_2}{3}m_{2|1}(z|f,g)+m_{2|\zeta_2}(z|f,g)+\\ &+\left(\mathcal{M}_{\kappa[1]}(z|\zeta_{[1]},\zeta_{[1]})_2+w_2(z)\right)\bar{f}\bar{g},\ \left(\mathcal{M}_{\kappa[1]}(z|\zeta_{[1]},\zeta_{[1]})_2+w_2(z)\right)\in D_{n_+}^{2k+1},\\ &M_{22|\mathrm{co}}(z|f,\mathfrak{I})=\{w_2(z),f(z)\},\ m_{22|\mathrm{co}}(f,g)=m_{22|\mathrm{co}}(f,\mathfrak{I})=0.\\ &M_{1D2}(z|f)=\zeta_{D2}(z|f),\ M_{12}(z|\mathfrak{I})=t_{D2}(z),\ m_{12}(f)=\mu_{12}(f),\ m_{12}(\mathfrak{I})=b_{22}.\end{aligned}$$

2.2.3 Higher orders

Introduce a form $\mathbf{N}_{2|\kappa,\zeta,w}(\mathbf{f},\mathbf{g})$,

$$\mathbf{N}_{2|\kappa,\zeta,w}(\mathbf{f},\mathbf{g}) = N_{2|\kappa,\zeta,w}(z|\mathbf{f},\mathbf{g}) + n_{2|\kappa,\zeta,w}(\mathbf{f},\mathbf{g})\mathcal{I},$$

where

$$\begin{array}{l}
N_{2|\kappa,\zeta,w}(z|f,g) = \mathcal{M}_\kappa(z|f+\zeta\bar{f}, g+\zeta\bar{g}) + w(z)\bar{f}\bar{g},\\
N_{2|\kappa,\zeta,w}(z|f,\mathcal{I}) = \mathcal{M}_\kappa(z|w, f+\zeta\bar{f}),\\
n_{2|\kappa,\zeta,w}(f,g) = \bar{f}\bar{g},\ n_{2|\kappa,\zeta,w}(f,\mathcal{I}) = 0,\ \mathbf{N}_{2|\kappa,\zeta,w}(z|\mathcal{I},\mathcal{I}) = 0,\\
\varepsilon(\zeta) = 1,\ \varepsilon(w) = 0,\\
\zeta(z), w(z) \in E^{2k+1}_{n_+}[[\hbar^2]]/D^{2k+1}_{n_+}[[\hbar^2]],\\
\psi = \mathcal{M}_\kappa(z|\zeta,\zeta) + w(z) \in D^{2k+1}_{n_+}[[\hbar^2]],
\end{array}$$

Note that it follows from the Jacoby identity for the form $\mathcal{M}_\kappa(z|f,g)$ that

$$\mathcal{M}_\kappa(z|\mathcal{M}_\kappa(|\zeta,\zeta),\zeta) \equiv 0,$$

such that

$$\begin{array}{rcl}
\mathcal{M}_{z\kappa}(|w,\zeta) & = & \mathcal{M}_\kappa(z|\mathcal{M}_\kappa(|\zeta,\zeta)+w,\zeta) = \mathcal{M}_\kappa(z|\psi,\zeta) \in D^{2k+1}_{n_+}[[\hbar^2]],\\
\bar{\mathcal{M}}_\kappa(z|w,\zeta) & = & \bar{\mathcal{M}}_\kappa(z|\psi,\zeta) = 0.
\end{array}$$

Analogously, the condition

$$\mathcal{M}_{\kappa_{[n]}}(z|\zeta_{[n]},\zeta_{[n]})_{[n+1]} + w(z)_{[n+1]} \in D^{2k+1}_{n_+}[[\hbar^2]],\ w(z)_{[n]} = \sum_{k=2}^{n}\hbar^{2k}w_k(z),$$

implies

$$\mathcal{M}_{\kappa_{[n+1]}}(z|w(z)_{[n+1]},\zeta_{[n+1]})_{[n+2]} \in D^{2k+1}_{n_+}[[\hbar^2]],\ \bar{\mathcal{M}}_{\kappa_{[n+1]}}(z|w(z)_{[n+1]},\zeta_{[n+1]})_{[n+2]} = 0.$$

Straightforward calculations give that the form $\mathbf{N}_{2|\kappa,\zeta,w}(\mathbf{f},\mathbf{g})$ satisfies the Jacoby identity

$$\begin{array}{l}
(-1)^{\varepsilon(\mathbf{f})\varepsilon(\mathbf{h})}\mathbf{N}_{2|\kappa,\zeta,w}(\mathbf{f},,\mathbf{N}_{2|\kappa,\zeta,w}(\mathbf{g},\mathbf{h})) + (-1)^{\varepsilon(\mathbf{h})\varepsilon(\mathbf{g})}\mathbf{N}_{2|\kappa,\zeta,w}(\mathbf{h},\mathbf{N}_{2|\kappa,\zeta,w}(\mathbf{f},\mathbf{g})) +\\
+(-1)^{\varepsilon(\mathbf{g})\varepsilon(\mathbf{f})}\mathbf{N}_{2|\kappa,\zeta,w}(\mathbf{g},\mathbf{N}_{2|\kappa,\zeta,w}(\mathbf{h},\mathbf{f})) = 0.
\end{array}$$

Performing the similarity transformation $[\mathbf{f},\mathbf{g}]_* \to [\mathbf{f},\mathbf{g}]_{*T}$ with $\mathbf{T}(\mathbf{f}) = \mathbf{f} - \hbar^4\mathbf{M}_{1D2}(\mathbf{f}) + O(\hbar^6)$, one can rewrite $[\mathbf{f},\mathbf{g}]_*$ in the form

$$\begin{array}{rcl}
[\mathbf{f},\mathbf{g}]_* & = & \mathbf{N}_{2|\kappa_{[2]},\zeta_{[2]},w_{[2]}}(\mathbf{f},\mathbf{g}) - \mathcal{L}_{23}(z|\mathbf{f},\mathbf{g}) + \hbar^6\mathbf{M}_{23}(\mathbf{f},\mathbf{g}) + O(\hbar^8),\\
\mathcal{L}_{23}(z|f,g) & = & \mathcal{M}_{\kappa_{[2]}}(z|\zeta_{[2]},\zeta_{[2]})_3\bar{f}\bar{g},\ \mathcal{L}_{23}(z|f,\mathcal{I}) = \mathcal{M}_{\kappa_{[2]}}(z|w_{[2]},\zeta_{[2]})_3\bar{f},\ \mathcal{L}_{23}(z|\mathcal{I},\mathcal{I}) = 0.
\end{array}$$

The condition $[\mathbf{f},\mathbf{g}]_* \in \mathbf{D}^{2k+1}_{n_+}[[\hbar^2]]$ implies that $\mathbf{M}_{23}(\mathbf{f},\mathbf{g}) \in \mathbf{D}^{2k+1}_{n_+}[[\hbar^2]]$.

In $\hbar^6$-order, Eq. (37) gives four equations. First two of them are

$$d_1^{\mathrm{ad}}\tilde{M}_{13}(z|f,g) = 0,\ \tilde{M}_{13}(z|f) = M'_{23}(z|f,\mathcal{I}) = M_{23}(z|f,\mathcal{I}) - \mathcal{M}_{\kappa_{[2]}}(z|w_{[2]},\zeta_{[2]})_3\bar{f}, \quad (43)$$

$$m_{23}(\{f,g\},\mathfrak{I}) = \bar{f}\bar{M}_{23}(|g,\mathfrak{I}) - (-1)^{\varepsilon(f)\varepsilon(g)}\bar{g}\bar{M}_{23}(|f,\mathfrak{I}). \tag{44}$$

It follows from Eq. (43) (with the properties $M_{23}(z|f,\mathfrak{I}) \in D^{2k+1}_{n_+}[[\hbar^2]]$, $\varepsilon_{M'_{23}} = 0$, taken into account)

$$M_{23}(z|f,\mathfrak{I}) = \mathcal{M}_{\kappa_{[2]}}(z|w_{[2]},\zeta_{[2]})_3\bar{f} + t^0_3\mathcal{E}_z f(z) + \{t_3(z), f(z)\}.$$

Eq. (44) transforms to

$$m_{23}(\{f,g\},\mathfrak{I}) = (2+n_+ - n_-)t^0_3]\bar{f}\bar{g},$$

and gives

$$t^0_3 = 0,\ m_{23}(f,\mathfrak{I}) = m_{23}\bar{f} = 0.$$

Now the third equation takes the form

$$\begin{aligned}&d^{\mathrm{ad}}_2 M'_{23}(z|f,g,h) = 0,\\ &M'_{23}(z|f,g,h) = M_{23}(z|f,g,h) - [\mathcal{M}_\kappa(z|\zeta_{[2]},\zeta_{[2]})_3 + t_3(z)]\bar{f}\bar{g}\end{aligned} \tag{45}$$

So, we have

$$\begin{aligned}&M_{23}(z|f,g) = \frac{2\kappa_1\kappa_3 + \kappa_2^2}{6}m_{2|1}(z|f,g) + m_{2|\zeta_3}(z|f,g) + \\ &+[\mathcal{M}_{\kappa_{[2]}}(z|\zeta_{[2]},\zeta_{[2]})_3 + t_3(z) + c_{23}]\bar{f}\bar{g} + d^{\mathrm{ad}}_1\zeta_{D3}(z|f,g),\\ &\mathcal{M}_{\kappa_{[2]}}(z|\zeta_{[2]},\zeta_{[2]})_3 + t_3(z) + c_{23} \in D^{2k+1}_{n_+}[[\hbar^2]],\\ &\varphi_{D3}(z|f) \in D^{2k+1}_{n_+}[[\hbar^2]],\ \zeta_3(z) \in E^{2k+1}_{n_+}[[\hbar^2]]/D^{2k+1}_{n_+}[[\hbar^2]].\end{aligned}$$

The last equation is

$$d^{\mathrm{tr}}_2 m'_{23}(f,g,h) = 0,\ m'_{23}(f,g,h) = m_{23}(f,g,h) - [\bar{\varphi}_{D3}(|f)\bar{g} - (-1)^{\varepsilon(f)\varepsilon(g)}\bar{\varphi}_{D3}(|g)\bar{f}],$$

and it implies

$$m_{23}(f,g) = \bar{\varphi}_{D3}(|f)\bar{g} - (-1)^{\varepsilon(f)\varepsilon(g)}\bar{\varphi}_{D3}(|g)\bar{f} - b_{23}\bar{f}\bar{g} - \mu_{13}(\{f,g\}).$$

Introduce a notation

$$t_3(z) + c_{23} = t_{D3}(z) + w_3(z),\ t_{D3}(z) \in D^{2k+1}_{n_+},\ w_3(z) \in E^{2k+1}_{n_+}[[\hbar^2]]/D^{2k+1}_{n_+}[[\hbar^2]].$$

Then, we can write

$$\mathbf{M}_{23}(\mathbf{f},\mathbf{g}) = \mathbf{M}_{23|\mathrm{co}}(\mathbf{f},\mathbf{g}) + \mathbf{d}^{\mathrm{ad}}_1\mathbf{M}_{1D3}(\mathbf{f},\mathbf{g}),$$

where

$$\begin{aligned}&M_{23|\mathrm{co}}(f,g) = \frac{2\kappa_1\kappa_3 + \kappa_2^2}{6}m_{2|1}(z|f,g) + m_{2|\zeta_3}(z|f,g) + \\ &+\left(\mathcal{M}_{\kappa_{[2]}}(z|\zeta_{[2]},\zeta_{[2]})_3 + w_3(z)\right)\bar{f}\bar{g},\ \left(\mathcal{M}_{\kappa_{[2]}}(z|\zeta_{[2]},\zeta_{[2]})_3 + w_3(z)\right) \in D^{2k+1}_{n_+}[[\hbar^2]],\end{aligned}$$

$$M_{23|\mathrm{co}}(z|f,\mathfrak{I}) = \{w_3(z), f(z)\} + \mathcal{M}_{\kappa_{[2]}}(z|w_{[2]},\zeta_{[2]})_3\bar{f},\ m_{23|\mathrm{co}}(f,g) = m_{23|\mathrm{co}}(f,\mathfrak{I}) = 0.$$

$$M_{1D3}(z|f) = \zeta_{D3}(z|f),\ M_{13}(z|\mathfrak{I}) = t_{D3}(z),\ m_{13}(f) = \mu_{13}(f),\ m_{13}(\mathfrak{I}) = b_{23}.$$

Performing the similarity transformation $[\mathbf{f},\mathbf{g}]_* \to [\mathbf{f},\mathbf{g}]_{*T}$ with $\mathbf{T}(\mathbf{f}) = \mathbf{f} - \hbar^6\mathbf{M}_{1D3}(\mathbf{f}) + O(\hbar^8)$, we rewrite $[\mathbf{f},\mathbf{g}]_*$ in the form

$$[\mathbf{f},\mathbf{g}]_* = \mathbf{N}_{2|\kappa_{[3]},\zeta_{[3]},w_{[3]}}(\mathbf{f},\mathbf{g}) + O(\hbar^8).$$

Proceeding in the same way, we finally find that up to similarity transformation, the general form of the deformation of the superalgebra $\mathbf{D}_{n_+}^{2k+1}$ is given by

$$\begin{aligned}
[\mathbf{f},\mathbf{g}]_* &= \mathbf{N}_{2|\kappa_{[\infty]},\zeta_{[\infty]},w_{[\infty]}}(\mathbf{f},\mathbf{g}),\\
\zeta_{[\infty]}(z), w_{[\infty]}(z) &\in E_{n_+}^{2k+1}/D_{n_+}^{2k+1},\ [\mathcal{M}_{\kappa_{[\infty]}}(z|\zeta_{[\infty]},\zeta_{[\infty]}) + w_{[\infty]}(z)] \in D_{n_+}^{2k+1}[[\hbar^2]].
\end{aligned}$$

3 Superalgebra $\mathbf{D}_{n_+}^{n_++4}$

In the case under consideration we have

$$\begin{aligned}
C(f,g) = \int du\,([\mathcal{E}_u f(u)]g(u) - f(u)\mathcal{E}_u g(u)) =\\
= 2\int du[\mathcal{E}_u f(u)]g(u),\ \mathcal{E}_u = 1 - \frac{1}{2}u^A\partial_A^u.
\end{aligned}$$

3.1 Second adjoint cohomology

We solve in this section the cohomology equation

$$\mathbf{d}_2^{\mathrm{ad}}\mathbf{M}_2(\mathbf{f},\mathbf{g},\mathbf{h}) = 0. \tag{46}$$

It follows from (17) that $d_1^{\mathrm{ad}}\tilde{M}_1(z|f,g) = 0$. The general solution of this equation has the form

$$M_2(z|f,\mathfrak{I}) = t^0\mathcal{E}_z f(z) + \{t'(z), f(z)\},$$

where

$$t'(z) \in E_{n_+}^{n_++4},\ \varepsilon(t^0) = \varepsilon(t'(z)) = \varepsilon_{M_2}.$$

Since $C(f,\mathcal{E}g)) - (-1)^{\varepsilon(f)\varepsilon(g)}C(g,\mathcal{E}f)) = 0$, we have

$$C(M_2(|f,\mathfrak{I}),g) - (-1)^{\varepsilon(f)\varepsilon(g)}C(M_2(|g,\mathfrak{I}),f) = C(t',\{f,g\})$$

It follows from (17) also that

$$m_2(\{f,g\},\mathfrak{I}) = C(t',\{f,g\})$$

which implies

$$m_2(f,\mathfrak{I}) = C(t',f) + m\bar{f}.$$

Further, it follows from (16) that

$$d_2^{\mathrm{ad}}M_2'(z|f,g,h) = -t^0[C(f,g)\mathcal{E}_z h(z) - (-1)^{\varepsilon(f)\varepsilon(g)}C(f,h)\mathcal{E}_z g(z) + C(g,h)\mathcal{E}_z f(z)], \tag{47}$$

where

$$M'_2(z|f,g,h) = M_2(z|f,g,h) - \tilde{C}_2(z|f,g),$$
$$\tilde{C}_2(z|f,g) = t'(z)C(f,g).$$

Here we used the relation

$$M_2(z|h,\mathfrak{I})C(f,g) - (-1)^{\varepsilon(f)\varepsilon(g)}M_2(z|g,\mathfrak{I})C(f,h) + M_2(z|f,\mathfrak{I})C(g,h) =$$
$$= t^0[\mathcal{E}_z h(z)C(f,g) - (-1)^{\varepsilon(f)\varepsilon(g)}\mathcal{E}_z g(z)C(f,h) + \mathcal{E}_z f(z)C(g,h)] - d_2^{\mathrm{ad}}\tilde{C}_2(z|f,g,h).$$

To solve Eq. (47), consider it in the following domains

1)

$$z\bigcap\left[\mathrm{supp}(f)\bigcup\mathrm{supp}(g)\bigcup\mathrm{supp}(h)\right] = \mathrm{supp}(f)\bigcap\left[\mathrm{supp}(g)\bigcup\mathrm{supp}(h)\right] = \varnothing.$$

2)

$$z\bigcap\left[\mathrm{supp}(f)\bigcup\mathrm{supp}(g)\right] = \mathrm{supp}(f)\bigcap\left[\mathrm{supp}(g)\bigcup\mathrm{supp}(h)\right] = \mathrm{supp}(g)\bigcap\mathrm{supp}(h) = \varnothing.$$

3)

$$z\bigcap\left[\mathrm{supp}(f)\bigcup\mathrm{supp}(g)\bigcup\mathrm{supp}(h)\right] = \varnothing.$$

4)

$$\left[z\bigcup\mathrm{supp}(f)\bigcup\mathrm{supp}(g)\right]\bigcap\mathrm{supp}(h) = \varnothing.$$

In each of these domains, the r.h.s. of Eq. (47) equals to zero, and we have (for details see [2])

$$M'_2(z|f,g) = \int du a(u)[\mathcal{E}_z f(z)g(u) - f(u)\mathcal{E}_z g(z)] +$$
$$+\int du\left(\{m^1(z|u), f(z)\}g(u) - (-1)^{\varepsilon(f)\varepsilon(g)}\{m^1(z|u), g(z)\}f(u)\right) +$$
$$+\mu(z)\int du\,([\mathcal{E}_u f(u)]g(u) - f(u)\mathcal{E}_u g(u)) + d_1^{\mathrm{ad}}\zeta^2(z|f,g) + M_{2|\mathrm{loc}}(z|f,g).$$

Let

$$\left[z\bigcup\mathrm{supp}(h)\right]\bigcap\left[\mathrm{supp}(f)\bigcup\mathrm{supp}(g)\right] = \varnothing$$

Then

$$(-1)^{\varepsilon(f)\varepsilon(h)}(-1)^{\varepsilon(g)\varepsilon(h)}(-1)^{\varepsilon(h)\varepsilon(M_2)}\{h(z), M_{2|2}(z|f,g)\} - \hat{M}_{2|2}(z|\{f,g\},h) = 0,$$

which implies

$$\{h(z),\mu(z)\}\int du\,([\mathcal{E}_u f(u)]g(u) - f(u)\mathcal{E}_u g(u)) + \mathcal{E}_z h(z)\int du a(u)\{f(u), g(u)\} +$$
$$+\int du\{\hat{m}^1(z|u), h(z)\}\{f(u), g(u)\} = t^0 C(f,g)\mathcal{E}_z h(z).$$

Choosing $h(z) = \mathrm{const}$ we obtain

$$\int dua(u)\{f(u), g(u)\} = -t^0 C(f,g) \Longrightarrow t^0 = 0.$$

So, we have

$$\begin{aligned}
M_2(z|f,g) &= c_1 m_{2|1}(z|f,g) + c_3 m_{2|3}(z|f,g) + t_D(z)C(f,g) + m_{2|\zeta}(z|f,g) + d_1^{\mathrm{ad}}\varphi_D(z|f,g),\\
t_D(z) &= t'(z) + c_4 \in D_{n_+}^{n_++4},\ \varepsilon(t_D(z)) = \varepsilon_{M_2},\\
\zeta(z) &\in E_{n_+}^{n_++4}/D_{n_+}^{n_++4},\ \varepsilon(\zeta(z)) = \varepsilon_{M_2},\ \varepsilon_{\varphi_D} = \varepsilon_{M_2},\\
M_2(z|f,\daleth) &= \{t_D(z), f(z)\},\\
m_2(f,\daleth) &= C(t_D, f) + m\bar{f},
\end{aligned}$$

where the bilinear forms $m_{2|1}$, $m_{2|3}$ and $m_{2|\zeta}$ are the same as in the Subsection 2.1.

To specify the parameters, use the relations

$$\begin{aligned}
&(-1)^{\varepsilon(g)\varepsilon(h)}C(m_{2|1}(|f,h),g) - C(m_{2|1}(|f,g),h) - (-)^{\varepsilon(f)\varepsilon(g)+\varepsilon(f)\varepsilon(h)}C(m_{2|1}(|g,h),f) =\\
&= -4\int du[f(u)\left(\overleftarrow{\partial}_A\omega^{AB}\partial_B\right)^3 g(u)]h(u),
\end{aligned}$$

$$(-1)^{\varepsilon(g)\varepsilon(h)}C(m_{2|3}(|f,h),g) - C(m_{2|3}(|f,g),h) - (-)^{\varepsilon(f)\varepsilon(g)+\varepsilon(f)\varepsilon(h)}C(m_{2|3}(|g,h),f) = 0, \quad (48)$$

$$\begin{aligned}
&(-1)^{\varepsilon(g)\varepsilon(h)}C(d_1^{\mathrm{ad}}\varphi(|f,h),g) - C(d_1^{\mathrm{ad}}\varphi(|f,g),h) - (-)^{\varepsilon(f)\varepsilon(g)+\varepsilon(f)\varepsilon(h)}C(d_1^{\mathrm{ad}}\varphi(|g,h),f) =\\
&= d_2^{\mathrm{tr}}[D_\zeta + C_{\varphi_D}](f,g,h),
\end{aligned}$$

where

$$\begin{aligned}
&D_\zeta(f,g) = C(\zeta,g)\bar{f} - (-1)^{\varepsilon(f)\varepsilon(g)}C(\zeta,f)\bar{g},\\
&C_{\varphi_D}(f,g) = C(\varphi_D(|f),g) - (-1)^{\varepsilon(f)\varepsilon(g)}C(\varphi_D(|g),f).
\end{aligned}$$

To obtain these relations, we used the following ones:

$$\begin{aligned}
&\int dzf(z)\mathcal{E}_z g(z) = -\int dz[\mathcal{E}_z f(z)]g(z),\quad n_- = n_+ + 4,\\
&C(\{t,f\},g) - (-1)^{\varepsilon(f)\varepsilon(g)}C(\{t,g\},f) = C(t,\{f,g\})
\end{aligned}$$

$$\int dzf(z)[g(z)\left(\overleftarrow{\partial}_A\omega^{AB}\partial_B\right)^p h(z)] = \int dz[f(z)\left(\overleftarrow{\partial}_A\omega^{AB}\partial_B\right)^p g(z)]h(z),\ p = 0,1,...,$$

$$\begin{aligned}
&\mathcal{E}_z[f(z)\left(\overleftarrow{\partial}_A\omega^{AB}\partial_B\right)^3 g(z)] - [\mathcal{E}_z f(z)]\left(\overleftarrow{\partial}_A\omega^{AB}\partial_B\right)^3 g(z) - f(z)\left(\overleftarrow{\partial}_A\omega^{AB}\partial_B\right)^3 \mathcal{E}_z g(z) =\\
&= 2f(z)\left(\overleftarrow{\partial}_A\omega^{AB}\partial_B\right)^3 g(z).
\end{aligned}$$

It follows from (16) also that

$$d_2^{\mathrm{tr}} m_2'(f,g,h) = m[C(g,h)\bar{f} - (-1)^{\varepsilon(f)\varepsilon(g)}C(f,h)\bar{g} + C(f,g)\bar{h}] -$$
$$-4c_1\int dz[f(z)\left(\overleftarrow{\partial}_A\omega^{AB}\partial_B\right)^3 g(z)]h(z),$$
$$m_2'(f,g) = m_2(f,g) - [D_\zeta + C_{\varphi_D}](f,g).$$

Let now

$$\left[\mathrm{supp}(f)\bigcup\mathrm{supp}(g)\right]\bigcap\mathrm{supp}(h) = \varnothing,\ g(z)=1,\ z\in\mathrm{supp}(f).$$

Then we have $mC(f,g)\bar{h} = -d_2^{\mathrm{tr}}\hat{m}_2'(f,g,h)$ and so $m=0$. In such a way

$$d_2^{\mathrm{tr}} m_2'(f,g,h) = -4c_1\int dz[f(z)\left(\overleftarrow{\partial}_A\omega^{AB}\partial_B\right)^3 g(z)]h(z).$$

Due to non-triviality of the cocycle $\int dz[f(z)\left(\overleftarrow{\partial}_A\omega^{AB}\partial_B\right)^3 g(z)]h(z) \in C_3(D^{n_-}_{n_+},\mathbb{K})$ (see [2]) we have $c_1 = 0$.

So, we obtain

$$m_2(f,g) = D_\zeta(f,g) + C_{\varphi_D}(f,g) - bC(f,g) - d_1^{\mathrm{tr}}\mu_1(f,g).$$

Finally, the general solution of Eq. (46) is

$$\begin{aligned}
\mathbf{M}_2(\mathbf{f},\mathbf{g}) &= \mathbf{M}_{2|1}(\mathbf{f},\mathbf{g}) + \mathbf{d}_1^{\mathrm{ad}}\mathbf{M}_{1D}(f,g),\\
M_{2|1}(f,g) &= c_3 m_{2|3}(z|f,g) + m_{2|\zeta}(z|f,g),\ m_{2|1}(f,g) = D_\zeta(f,g),\\
\mathbf{M}_{2|1}(\mathbf{f},\mathfrak{I}) &= 0,\\
M_{1D}(z|f) &= \varphi_D(z|f),\ M_1(z|\mathfrak{I}) = -t_D(z),\ m_1(f) = \mu_1(f),\ m_1(\mathfrak{I}) = b.
\end{aligned}$$

3.1.1 Independence and non-triviality

Let us suppose that

$$\mathbf{M}_{2|1}(\mathbf{f},\mathbf{g}) = \mathbf{d}_1^{\mathrm{ad}}\mathbf{M}_1(f,g).$$

It follows from this equation that

$$c_3 m_{2|3}(z|f,g) = d_1^{\mathrm{ad}} M_1'(z|f,g) - C(f,g)M_1(\mathfrak{I}) \tag{49}$$

for some $M_1'(z|f)$. Let

$$z\bigcap\left[\mathrm{supp}(f)\bigcup\mathrm{supp}(g)\right] = \varnothing,\ g(z)=1,\ z\in\mathrm{supp}(f).$$

It follows from Eq. (49)

$$\hat{M}_1'(z|\{f,g\}) + C(f,g)M_1(\mathfrak{I}) = 0 \implies M_1(\mathfrak{I}) = 0,$$

but then Eq. (49) has solutions for $c_3 = 0$ only.

3.2 Deformations of Lie superalgebra $\mathbf{D}_{n_+}^{n_+ +4}$

In this section, we find the general form of the deformation of Lie superalgebra $\mathbf{D}_{n_+}^{n_+ +4}$, $[f,g]_*$,

$$\begin{aligned}[\mathbf{f},\mathbf{g}]_* = &\ M_2^*(z|\mathbf{f},\mathbf{g}) + m_2^*(\mathbf{f},\mathbf{g})\mathfrak{I} = \sum_{l=0}^{\infty} \hbar^{2l}\mathbf{M}_{2l}(\mathbf{f},\mathbf{g}),\\ &\ M_2^*(z|\mathbf{f},\mathbf{g}) \in D_{n_+}^{n_+ +4}[[\hbar^2]],\ m_2^*(\mathbf{f},\mathbf{g}) \in \mathbb{K}[[\hbar^2]],\ \varepsilon_{M_2^*} = \varepsilon_{m_2^*} = 0,\end{aligned}$$

satisfying the Jacoby identity.

3.2.1 $\hbar^0$-order

We have

$$M_{20}(f,g) = \{f(z),g(z)\}\,,\ m_{20}(f,g) = C(f,g),\ M_{20}(\mathbf{f},\mathfrak{I}) = m_{20}(\mathbf{f},\mathfrak{I}) = 0.$$

3.2.2 $\hbar^2$-order

In $\hbar^2$-order, Jacoby identity gives the equation

$$\mathbf{d}_2^{\mathrm{ad}}\mathbf{M}_{21}(f,g,h) = 0$$

The general solution of this equation is found in the preceding section

$$\mathbf{M}_{21}(\mathbf{f},\mathbf{g}) = \mathbf{M}_{2|11}(\mathbf{f},\mathbf{g}) + \mathbf{d}_1^{\mathrm{ad}}\mathbf{M}_{1D1}(f,g),$$

$$\begin{aligned}&M_{2|11}(f,g) = c_3 m_{2|3}(z|f,g) + m_{2|\zeta_1}(z|f,g),\ \varepsilon(\zeta_1) = 0,\\ &m_{2|11}(f,g) = D_{\zeta_1}(f,g),\\ &\mathbf{M}_{2|11}(\mathbf{f},\mathfrak{I}) = 0\\ &\mathbf{M}_{1D1}(f) = \varphi_{D1}(z|f) + m_{11}(f)\mathfrak{I},\ \mathbf{M}_{1D1}(\mathfrak{I}) = M_{1D1}(z|\mathfrak{I}) + b_1\mathfrak{I}.\end{aligned}$$

3.2.3 Higher orders

Introduce a form

$$\begin{aligned}\mathbf{S}_{2|\zeta,c_3}(\mathbf{f},\mathbf{g}) &= S_{2|\zeta,c_3}(z|\mathbf{f},\mathbf{g}) + s_{2|\zeta,c_3}(\mathbf{f},\mathbf{g})\mathfrak{I},\\ S_{2|\zeta,c_3}(z|f,g) &= \{f(z),g(z)\} + c_3 m_{2|3}(z|f,g) + m_{2|\zeta}(z|f,g),\\ s_{2|\zeta,c_3}(f,g) = C(f,g) + D_\zeta(f,g),&\\ S_{2|\zeta,c_3}(\mathbf{f},\mathfrak{I}) &= s_{2|\zeta,c_3}(\mathbf{f},\mathfrak{I}) = 0.\end{aligned}$$

The form $\mathbf{S}_{2|\zeta,c_3}(z|\mathbf{f},\mathbf{g})$ satisfy the Jacoby identity

$$(-1)^{\varepsilon(\mathbf{f})\varepsilon(\mathbf{h})}\mathbf{S}_{2|\zeta,c_3}(\mathbf{S}_{2|\zeta,c_3}(\mathbf{f},\mathbf{g}),\mathbf{h}) + \mathrm{cycle}(\mathbf{f},\mathbf{g},\mathbf{h}) = 0.$$

To prove this fact, one should use the fact that the form $S_{2|\zeta,c_3}(z|f,g)$ satisfy the Jacoby identity (see [3]), the relation (48), and relations

$$(-1)^{\varepsilon(f)\varepsilon(h)}C(\{f,g\},h) + \mathrm{cycle}(f,g,h) = (-1)^{\varepsilon(f)\varepsilon(h)}d_2^{\mathrm{tr}}C(f,g,h) = 0,$$

$$(-1)^{\varepsilon(f)\varepsilon(h)}C(m_{2|\zeta}(|f,g),h) + \text{cycle}(f,g,h) = -(-1)^{\varepsilon(f)\varepsilon(h)}d_2^{\text{tr}}D_\zeta(f,g,h),$$
$$(-1)^{\varepsilon(f)\varepsilon(h)}D_\zeta(\{f,g\},h) + \text{cycle}(f,g,h) = (-1)^{\varepsilon(f)\varepsilon(h)}d_2^{\text{tr}}D_\zeta f,g,h),$$
$$(-1)^{\varepsilon(f)\varepsilon(h)}C(m_{2|\zeta}(|f,g),h) + (-1)^{\varepsilon(f)\varepsilon(h)}D_\zeta(\{f,g\},h) + \text{cycle}(f,g,h) = 0,$$

$$\begin{aligned} (-1)^{\varepsilon(f)\varepsilon(h)}D_\zeta(m_{2|3}(|f,g),h) + \text{cycle}(f,g,h) &= 0, \\ (-1)^{\varepsilon(f)\varepsilon(h)}D_\zeta(m_{2|\zeta}(|f,g),h) + \text{cycle}(f,g,h) &= 0. \end{aligned}$$

Performing the similarity transformation $[\mathbf{f},\mathbf{g}]_* \to [\mathbf{f},\mathbf{g}]_{*T}$ with $\mathbf{T}(\mathbf{f}) = \mathbf{f} - \hbar^2\mathbf{M}_{11}(\mathbf{f}) + O(\hbar^4)$, one can rewrite $[\mathbf{f},\mathbf{g}]_*$ in the form

$$[\mathbf{f},\mathbf{g}]_* = \mathbf{S}_{2|\zeta_{[1]},c_{3[1]}}(\mathbf{f},\mathbf{g}) + \hbar^4\mathbf{M}_{22}(\mathbf{f},\mathbf{g}) + O(\hbar^6).$$

In $\hbar^4$-order, the Jacoby identity gives the equation

$$\mathbf{d}_2\mathbf{M}_{22}(\mathbf{f},\mathbf{g}) = 0.$$

Proceeding in the same way, we finally find that up to a similarity transformation, the general form of the deformation of the centrally extended super Poincare algebra is given by

$$[\mathbf{f},\mathbf{g}]_* = \mathbf{S}_{2|\zeta_{[\infty]},c_{3[\infty]}}(\mathbf{f},\mathbf{g}).$$

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